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Geometrical factor and thermal properties of the Potts model

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Abstract. Based on the connection between the q -state Potts model (QPM) and a q -state bond-correlated percolation model (QBCPM), we define a non-percolating geometrical factor g_f and percolating geometrical factor g_p which depend only on q and geometrical properties of the system. The thermal properties of the QPM could be derived simply from g_f and g_p . The idea is confirmed by an exact calculation for one-dimensional QPM and may be extended to other Ising-like spin models. Our formulation also provides a geometrical meaning for the finite-size scaling and broadening at first-order phase transitions of the spin model.

Since Onsager (1944) published the exact solution of the spin- $\frac{1}{2}$ simple Ising model on a square lattice, many Ising-like spin models, e.g. the q -state Potts model (Potts 1952, Wu 1982), have been proposed and studied. Since Hammersley (1957) and Broadbent and Hammersley (1957) advocated the concept of percolation, percolation has become an important branch of statistical mechanics (Essam 1973, 1980, Stauffer 1979, Stauffer *et al* 1982, Deutscher *et al* 1983). It has been found that the phase transition in the Ising-like spin model and the percolation transition in the lattice percolation model have many characteristics in common. The singular behaviour in the latter is clearly related to the onset of the appearance of the percolating cluster in the system. Since Müller-Krumbhaar (1974) first studied the possible connection between the phase transition in the Ising model and the percolation transition of a correlated percolation model, much effort (for reviews see Stauffer 1979, Stauffer *et al* 1982, Kertész *et al* 1983, Hu 1984a) has been devoted to studying whether the phase transition in the Ising model has a mechanism similar to that of the percolation transition, i.e. the onset of the appearance of the percolating cluster. Using a new idea to define 'clusters' for Ising-like spin models, recently Hu (1983a, b, 1984a, b, c, d, e, 1985a, b)† has formally shown that the partition functions of many Ising-like spin models are the generating functions of certain correlated percolation or cluster models which have the same critical properties as the corresponding spin models. Thus in Hu's picture of 'clusters', the phase transitions in many Ising-like spin models are indeed the percolation transition and the physical quantities of the spin models are related to the geometrical quantities of the corresponding percolation models.

It is of interest to know whether such connections could help us to understand more clearly, or to calculate more easily, the thermal properties of the spin model. Based on the connection (Hu 1984c, d) between the q -state Potts model (QPM) and a q -state bond-correlated percolation model (QBCPM), we could define a non-percolating

† The reader is advised to read Hu (1984a) first.

geometrical factor g_r and percolating geometrical factor g_p which depend only on the geometrical properties of the system. In this paper we propose that the thermal properties of the QPM could be deduced simply from the logarithm of such geometrical factors and their derivatives. Our idea is confirmed by an exact calculation for one-dimensional QPM and could be extended to other Ising-like spin models.

Suppose we have a lattice G of E nearest-neighbour (NN) bonds and N sites labelled by $i = 1, 2, \dots, N$. To each lattice site, we assign a q -state Potts spin s_i which has spin components $j, j-1, \dots, -(j-1), -j$, where $2j+1 = q$ and q is a positive integer. The partition function of the q -state Potts model (QPM) on G may be written as†

$$\begin{aligned}
 Z_N &= \sum_{s_i=-j}^j \prod_{\langle ij \rangle} \exp[K\delta(s_i, s_j)] \prod_i \exp(Bs_i) \\
 &= \sum_{s_i=-j}^j \prod_{\langle ij \rangle} [1 + (e^K - 1)\delta(s_i, s_j)] \prod_i \exp(Bs_i). \tag{1}
 \end{aligned}$$

Here the first product and the second product are over all NN bonds and sites of G , respectively, $K = \beta J$, $B = \beta h$; $\delta(s_i, s_j) = 1$ when $s_i = s_j$ and $\delta(s_i, s_j) = 0$ when $s_i \neq s_j$. Note that the coupling of B with s_i in (1) is different from that of Wu (1982). Now we expand the first product in (1) and use the subgraphs $G' \subseteq G$ to represent the terms in the expansion. For each NN bond $\langle ij \rangle$ there occurs in (1) the two terms 1 and $(e^K - 1)\delta(s_i, s_j)$; subgraphs G' with no $\langle ij \rangle$ bond correspond to the former and those with an $\langle ij \rangle$ bond to the latter. There are $b(G')$ bonds in the subgraph G' , $0 \leq b(G') \leq E$. If a particular bond $\langle ij \rangle$ is attached by the factor $(e^K - 1)\delta(s_i, s_j)$, then $s_i = s_j$ after summing over spin states and the sites i and j are said to be in the same cluster.

In general, if two sites can be connected through a series of bonds, they are said to be in the same cluster. The cluster which extends from one side of G to the opposite side of G is called the percolating cluster; otherwise, it is called the non-percolating or finite cluster. The percolating cluster becomes an infinite cluster in the limit $N \rightarrow \infty$. A given G' usually contains a large number of independent clusters including isolated sites which do not connect with any other sites via bonds. For a given G' , we can sum over all spin states and in such a summation only the terms where all spins in the same cluster have the same spin component have non-zero contributions. However, spins in different clusters of G' could have different spin components. After summing over spin states, we have‡

$$\begin{aligned}
 Z_N &= \sum_{G' \subseteq G} (e^K - 1)^{b(G')} \prod_c \{ \exp(Bn_c j) + \exp[Bn_c(j-1)] + \dots \\
 &\quad + \exp(-Bn_c j) \} \\
 &= e^{KE} \sum_{G' \subseteq G} p^{b(G')} (1-p)^{E-b(G')} \prod_c Z_c(B) \tag{2a}
 \end{aligned}$$

where \prod_c extends over all clusters c in G' , $n_c \equiv n_c(G')$ is the number of sites in the cluster c ,

$$p = 1 - e^{-K} \tag{2b}$$

$$Z_c(B) = \exp(Bn_c j) + \exp[Bn_c(j-1)] + \dots + \exp(-Bn_c j). \tag{2c}$$

† Do not confuse the spin component 'j' with the spin index in 's_j'.

‡ The following expansion for $B = 0$ has been derived by other authors. Wu (1982) considered the coupling of QPM and B in a different way. See the discussions at the end of Hu (1984a, b).

Note that $Z_c(B)$ is the partition function for a single equivalent spin with moment n_c . The magnetic moment of the system is given by

$$\begin{aligned} \frac{\partial}{\partial B} \ln Z_N &= \frac{1}{Z_N} \sum_{G' \subseteq G} (e^K - 1)^{b(G')} \prod_{c'} Z_{c'}(B) \sum_c \frac{\partial}{\partial B} \ln Z_c(B) \\ &= e^{KE} e^{BNj} \frac{1}{Z_N} \sum_{G' \subseteq G} P^{b(G')} (1-p)^{E-b(G')} \prod_{c'} \bar{Z}(Bn_{c'}) \sum_c n_c j B_j(Bn_c) \end{aligned} \tag{3a}$$

where

$$\bar{Z}(x) = 1 + e^{-x} + \dots + e^{-2jx} \rightarrow \begin{cases} 1 & \text{as } x \rightarrow \infty \\ q & \text{as } x \rightarrow 0 \end{cases} \tag{3b}$$

and where $B_j(x)$ is the Brillouin function (Reif 1965)

$$B_j(x) = j^{-1} [(j + \frac{1}{2}) \coth(j + \frac{1}{2})x - \frac{1}{2} \coth \frac{1}{2}x] \rightarrow \begin{cases} 1 & \text{as } x \rightarrow \infty \\ 0 & \text{as } x \rightarrow 0. \end{cases} \tag{3c}$$

$B_j(x)$ determines the magnetic moment of a paramagnetic system (Reif 1965).

The spontaneous magnetisation is defined by

$$M(G, p) = \lim_{B \rightarrow 0^+} \lim_{N \rightarrow \infty} \frac{\partial}{\partial B} (\ln Z_N) / N.$$

If for given G' , $n_c(N)$ is the number of sites in cluster c for a sample of size N then, as $N \rightarrow \infty$, if c is a non-percolating (finite) cluster $n_c(N) \rightarrow n_c < \infty$ but if c is a percolating (infinite) cluster $n_c(N) \rightarrow \infty$. Thus

$$\lim_{B \rightarrow 0^+} \lim_{N \rightarrow \infty} B_j(Bn_c) \prod_{c'} \bar{Z}(Bn_{c'}) = \begin{cases} 0 & \text{if } c \text{ is finite} \\ q^{n_c(G')} & \text{if } c \text{ is infinite} \end{cases}$$

and

$$\begin{aligned} M(G, P) &= j \lim_{N \rightarrow \infty} W^{-1} \sum_{G' \subseteq G} \pi(G', p, q) [N^*(G') / N] \\ &\equiv j \langle N^*(G') \rangle_0. \end{aligned} \tag{4}$$

The internal energy U and the specific heat C_h of the QPM in the thermodynamic limit $N \rightarrow \infty$ and for $B \rightarrow 0^+$ are given by

$$\begin{aligned} U(G, p) &= \lim_{B \rightarrow 0^+} \lim_{N \rightarrow \infty} \frac{-\partial}{\partial \beta} (\ln Z_N) / N \\ &= -\frac{J}{p} \langle b(G') \rangle_0 = -\frac{zJ}{2p} \bar{p} \end{aligned} \tag{5}$$

$$\begin{aligned} C_h(G, p) &= \frac{\partial}{\partial T} U(G, p) \\ &= \frac{kK^2}{p^2} [-(1-p) \langle b(G') \rangle_0 + \langle (b(G') - [b(G')]_{av})^2 \rangle_0] \\ &\equiv \frac{kK^2}{p^2} [-\frac{1}{2}z(1-p)\bar{p} + \langle (\delta b(G'))^2 \rangle_0]. \end{aligned} \tag{6}$$

In (4)–(6)

$$\pi(G', p, q) = p^{b(G')} (1-p)^{E-b(G')} q^{n_f(G')} \tag{7}$$

$$W = \sum_{G' \subseteq G} \pi(G', p, q) \tag{8}$$

$$\langle N^*(G') \rangle_0 = \lim_{N \rightarrow \infty} W^{-1} \sum_{G' \subseteq G} \pi(G', p, q) [N^*(G')/N] \tag{9}$$

$$\bar{p} = \lim_{N \rightarrow \infty} W^{-1} \sum_{G' \subseteq G} \pi(G', p, q) (b(G')/E) \tag{10}$$

$$[b(G')]_{av} = W^{-1} \sum_{G' \subseteq G} \pi(G', p, q) b(G') \tag{11}$$

$$\delta b(G') = b(G') - [b(G')]_{av} \tag{12}$$

and we have used $\langle Q(G') \rangle_0$ to represent the mean value of a subgraph-dependent quantity $Q(G')$ per site in the limit $N \rightarrow \infty$ (see also (9)). $N^*(G')$ in (4) is the total number of sites in the percolating clusters of G , $n_f(G')$ in (7) is the total number of non-percolating (i.e. finite) clusters in G' and z in (5) and (6) is the coordination number of G . It is obvious that $M(G, p)/j$ is the percolation probability of the following q -state bond-correlated percolation model (QBCPM) on G .

(1) All sites of G are occupied and each bond of G is attached with the bond probability $p = 1 - e^{-K}$.

(2) The overall probability of $G' \subseteq G$ is enhanced by a factor q for each finite cluster in G' . Equations (5) and (6) relate U and C_h of the QPM to the average number of occupied bonds, \bar{p} , and to the fluctuations of the number of occupied bonds $\langle (\delta b(G'))^2 \rangle_0$, for the QBCPM. Thus Z_N of (2) is the generating function of the QBCPM.

Now we classify all subgraphs $G' \subseteq G$ into subgraphs with only non-percolating clusters, G'_f , and subgraphs with at least one percolating cluster, G'_p , and consider the partial sums†

$$W_f = e^{KE} \sum_{G'_f \subseteq G} \pi(G'_f, p, q) = \sum_{G'_f \subseteq G} (e^K - 1)^{b(G'_f)} q^{n_f(G'_f)} \tag{13}$$

$$W_p = e^{KE} \sum_{G'_p \subseteq G} \pi(G'_p, p, q) = \sum_{G'_p \subseteq G} (e^K - 1)^{b(G'_p)} q^{n_f(G'_p)}. \tag{14}$$

Since only subgraphs G'_p contribute to $M(G, p)$, which is zero for $T > T_c$ and non-zero for $T < T_c$, it is reasonable to expect that for $p < p_c = 1 - e^{-K_c} = 1 - e^{-J/kT_c}$,

$$W_f \gg W_p \tag{15}$$

for $p > p_c$,

$$W_f \ll W_p \tag{16}$$

and at p_c

$$W_f = W_p. \tag{17}$$

Equation (17) is called the geometrical condition of phase transitions.

† From now on, we consider only $B \rightarrow 0^+$.

In the following, we will consider fixed but very large E . W_f and W_p may be represented as follows:

$$\begin{aligned}
 W_f &= \sum_{b=0}^E (e^K - 1)^{RE} \sum_{G'_f \subseteq G} q^{n_f(G'_f)} \\
 &= E \int_0^1 I(K, R) g_f(q, R) dR \tag{18}
 \end{aligned}$$

$$\begin{aligned}
 W_p &= \sum_{b=0}^E (e^K - 1)^{RE} \sum_{G'_p \subseteq G} q^{n_p(G'_p)} \\
 &= E \int_0^1 I(K, R) g_p(q, R) dR. \tag{19}
 \end{aligned}$$

Here

$$0 \leq R = b(G')/E \leq 1 \tag{20}$$

$$I(K, R) = (e^K - 1)^{RE} \tag{21}$$

$$g_f(q, R) = \sum_{G'_f \subseteq G} q^{n_f(G'_f)} \tag{22}$$

$$g_p(q, R) = \sum_{G'_p \subseteq G} q^{n_p(G'_p)}. \tag{23}$$

The summations in (22) and (23) are over all G'_f and G'_p , respectively, with a fixed fraction of occupied bonds R . Since g_f and g_p are independent of $K = J/kT$, they will be called the non-percolating geometrical factor and percolating geometrical factor, respectively. The dependence of W_f and W_p on K comes only from $I(K, R)$ and we will call $I(K, R)$ the interaction factor.

For a fixed K , $I(K, R)$ is a monotonous function of R , but $g_f(q, R)$ and $g_p(q, R)$ are expected to peak sharply at certain values. It is reasonable to expect that the integrands in (18) and (19) also peak sharply at certain values, say R_f and R_p respectively, which are solutions of the equations

$$\frac{d}{dR} [\ln I(K, R) g_f(q, R)]_{R_f} = E \ln(e^K - 1) + \frac{d}{dR} [\ln g_f(q, R)]_{R_f} = 0 \tag{24}$$

$$\frac{d}{dR} [\ln I(K, R) g_p(q, R)]_{R_p} = E \ln(e^K - 1) + \frac{d}{dR} [\ln g_p(q, R)]_{R_p} = 0. \tag{25}$$

Expanding the logarithm of the integrands in (18) and (19) at R_f and R_p , respectively, and keeping only the leading and the quadratic terms, we have

$$\begin{aligned}
 W_f &\approx EI(K, R_f) g_f(q, R_f) \int_0^1 \exp[-(R - R_f)^2/\Gamma_f^2] dR \\
 &\equiv A_f \Gamma_f \int_{-y_f}^{y_f} \exp(-y^2) dy \tag{26}
 \end{aligned}$$

$$\begin{aligned}
 W_p &\approx EI(K, R_p) g_p(q, R_p) \int_0^1 \exp[-(R - R_p)^2/\Gamma_p^2] dR \\
 &\equiv A_p \Gamma_p \int_{-y_p}^{y_p} \exp(-y^2) dy \tag{27}
 \end{aligned}$$

where

$$\Gamma_f^{-2} = -\frac{1}{2} \left(\frac{d^2}{dR^2} \ln g_f(q, R) \right)_{R_f} \tag{28}$$

$$\Gamma_p^{-2} = -\frac{1}{2} \left(\frac{d^2}{dR^2} \ln g_p(q, R) \right)_{R_p} \tag{29}$$

$$A_f = EI(K, R_f)g_f(q, R_f) \tag{30}$$

$$A_p = EI(K, R_p)g_p(q, R_p) \tag{31}$$

$$y_f = R_f/\Gamma_f \quad \bar{y}_f = (1 - R_f)/\Gamma_f \tag{32}$$

$$y_p = R_p/\Gamma_p \quad \bar{y}_p = (1 - R_p)/\Gamma_p. \tag{33}$$

In the very large E limit, we expect that Γ_f and Γ_p , the half-widths of the peaks at R_f and R_p , respectively, tend to zero and $y_f \bar{y}_f$, y_p and \bar{y}_p tend to ∞ . The integrations in (26) and (27) simply give the constant $\sqrt{\pi}$.

Thus

$$W_f \approx A_f \Gamma_f \int_{-\infty}^{\infty} \exp(-y^2) dy = A_f \Gamma_f \sqrt{\pi} \equiv W_f^* \tag{34}$$

$$W_p \approx A_p \Gamma_p \int_{-\infty}^{\infty} \exp(-y^2) dy = A_p \Gamma_p \sqrt{\pi} \equiv W_p^*. \tag{35}$$

The free energy per spin is given by

$$f(G, q, K) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln(W_f^* + W_p^*). \tag{36}$$

For a geometrical quantity $Q(R)$ of the QBCPM which depends on R , we may use the representations of (18) and (19) to calculate the average value of $Q(R)$:

$$[Q(R)]_{av} = \frac{\int_0^1 I(K, R)g_f(q, R)Q(R) dR + \int_0^1 I(K, R)g_p(q, R)Q(R) dR}{\int_0^1 I(K, R)g_f(q, R) dR + \int_0^1 I(K, R)g_p(q, R) dR}. \tag{37}$$

If $Q(R)$ is a sufficiently smooth function of R , e.g. a low degree polynomial, the first and second integrands in the numerator of (37) still peak sharply at R_f and R_p , respectively. We may proceed as before to carry out the integrations.

For example, the average number of occupied bonds (per bond), \bar{p} , is given by

$$\begin{aligned} \bar{p} &= \frac{[ER]_{av}}{E} = \frac{A_f \int_0^1 \exp[-(R - R_f)^2/\Gamma_f^2] R dR + A_p \int_0^1 \exp[-(R - R_p)^2/\Gamma_p^2] R dR}{A_f \int_0^1 \exp[-(R - R_f)^2/\Gamma_f^2] dR + A_p \int_0^1 \exp[-(R - R_p)^2/\Gamma_p^2] dR} \\ &= \frac{W_f^* R_f + W_p^* R_p}{W_f^* + W_p^*}. \end{aligned} \tag{38}$$

The fluctuation of the number of bonds is given by

$$\begin{aligned} \langle (\delta b(G'))^2 \rangle_0 &\equiv \frac{[(ER - E\bar{p})^2]_{av}}{N} \\ &= \frac{A_f E^2 \int_0^1 \exp[-(R - R_f)^2/\Gamma_f^2] (R - \bar{p})^2 dR + A_p E^2 \int_0^1 \exp[-(R - R_p)^2/\Gamma_p^2] (R - \bar{p})^2 dR}{N(W_f^* + W_p^*)} \\ &= \frac{E^2}{N(W_f^* + W_p^*)} \left[\frac{1}{2} W_f^* \Gamma_f^2 + \frac{1}{2} W_p^* \Gamma_p^2 + (R_f - \bar{p})^2 W_f^* + (R_p - \bar{p})^2 W_p^* \right]. \end{aligned} \tag{39}$$

For $T > T_c$, $W_f^* \gg W_p^*$ and we have

$$\bar{p} = \bar{R}_f \tag{40}$$

$$\langle (\delta b(G'))^2 \rangle_0 = \frac{1}{4} z E \Gamma_f^2. \tag{41}$$

For $T < T_c$, $W_f^* \ll W_p^*$ and we have

$$\bar{p} = \bar{R}_p \tag{42}$$

$$\langle (\delta b(G'))^2 \rangle_0 = \frac{1}{4} z E \Gamma_p^2. \tag{43}$$

At $T = T_c$, $W_f^* = W_p^*$ and we have

$$\bar{p} = \frac{1}{2} (\bar{R}_f + \bar{R}_p) \tag{44}$$

$$\langle (\delta b(G'))^2 \rangle_0 = \frac{E^2}{4N} [\Gamma_f^2 + \Gamma_p^2 + (R_p - R_f)^2]. \tag{45}$$

Assuming at T_c

$$ER_p - ER_f = b_0 N^a \tag{46}$$

where a and b are constants and $0 \leq a \leq 1$, we have

$$\langle (\delta b(G'))^2 \rangle_0 = \frac{E^2}{4N} (\Gamma_f^2 + \Gamma_p^2) + \frac{1}{4} b_0^2 N^{2a-1}. \tag{47}$$

The second term of (47) gives the singular part of the specific heat C_h^* which is

$$C_h^* \sim N^{\alpha/(2-\alpha)} \tag{48}$$

from the finite-size scaling of the specific heat at second-order phase transitions (Barber 1983). Identifying $2a - 1$ with $\alpha/(2 - \alpha)$, we have

$$\alpha = \frac{2a - 1}{a} = 2 - \frac{1}{a} \tag{49}$$

i.e. α increases with a .

To demonstrate that we may actually calculate the physical quantity of the QPM from the geometrical factor of the QBCCPM, we now carry out an exact calculation for the one-dimensional system with periodic boundary conditions. In such a system, $E = N$, $z = 2$, $g_p(q, R) = 0$ for $0 \leq R < 1$ and $g_f(q, 1) = 0$.

It follows from the Euler theorem that for any subgraph G' with $b(G') = NR < 1$, the number of clusters is given by $n_f(G') = N - b(G')$. Thus for a fixed $R = b(G')/N$, $n_f(G')$ is also fixed and the summation in (22) is reduced to the problem of counting the number of independent subgraphs for a fixed number of occupied bonds. Therefore

$$g_f(q, R) = C_{RN}^N q^{N-RN} = \frac{N!}{(N - RN)! RN!} q^{N(1-R)} \tag{50}$$

for $R < 1$. Using the left-hand side of (50) in (24) and (28), we find that in the large E limit

$$R_f = \frac{e^K - 1}{e^K + q - 1} \tag{51}$$

$$\Gamma_f^2 E = 2R_f(1 - R_f) = \frac{2q(e^K - 1)}{(e^K + q - 1)^2}. \tag{52}$$

Thus Γ_f goes to zero as $1/\sqrt{E}$.

It follows from (36), (40), (41), (5) and (6) that

$$f(G, q, K) = \ln(e^K + q - 1) \quad (53)$$

$$\bar{p} = R_f = \frac{e^K - 1}{e^K + q - 1} \quad (54)$$

$$\langle (\delta b(G'))_0 \rangle = \frac{1}{2} \Gamma_f^2 E = \frac{q(e^K - 1)}{(e^K + q - 1)^2} \quad (55)$$

$$U(G, p) = -\frac{J}{p} \bar{p} = -\frac{J e^K}{e^K + q - 1} \quad (56)$$

$$C_h(G, p) = \frac{kK^2 e^K (q - 1)}{(e^K + q - 1)^2}. \quad (57)$$

The results of (53), (56) and (57) are identical with those obtained from the standard transfer matrix method where the largest eigenvalue of the transfer matrix is $e^K + q - 1$ (Hu and Lee 1984).

For systems in higher dimensions, it is not easy to calculate g_f and g_p exactly. However, one may use the Monte Carlo simulation method to calculate the shapes of g_f and g_p and the ratio g_f/g_p from which one may obtain U , C_h , T_c and α . However, the calculation of $f(G, q, K)$ requires the absolute magnitudes of g_f and g_p which are not easily available from Monte Carlo simulations for systems with $q \neq 1$.

Equations (46) and (49) may be used to explain the change in the phase transition behaviour of the QPM on dimensions $d \geq 2$ when q is increased (Hu 1984c, d).

It follows from (7) that the subgraphs with larger numbers of finite clusters increase their relative probability weight when q is increased. For a given $R < 1$ we expect that the subgraphs G'_f contributing to the summation of (22) are usually more compact, and hence have more closed loops and finite clusters (Euler relations), than the subgraphs contributing to the summation of (23). Usually, the smaller the value of R the larger the number $n_f(G')$ in G' . Therefore when q is increased g_f will increase faster than g_p and the peak of g_f will move toward smaller values of R with a speed faster than that of g_p . After increasing q to a new value, we must also increase K to a new value so that (17), (24) and (25) may still be true. At the new K , the separation between R_p and R_f is expected to increase, i.e. a of (46) will increase with q . It follows from (49) that α will increase with q . When q is larger than a critical value q_c , a of (46) will assume its largest value 1 and we have

$$E(R_p - R_f) = b_0 N \quad (58)$$

$$C_h^* = \frac{kK^2}{4p^2} b_0^2 N = \frac{l^2}{4kT_c^2} N \quad (59)$$

$$l = U^+ - U^- = (J/p) b_0 \quad (60)$$

where U^+ and U^- are internal energies at $T_c + \varepsilon$ and $T_c - \varepsilon$, respectively, with ε being an infinitesimal positive number. Thus for $q > q_c$, the latent heat $l > 0$ and the phase transition is first order. Equation (59) is the finite-size scaling of specific heat at a first-order phase transition (Barber 1983, Hu and Kleban 1981, 1983, Kleban and Hu 1982). The finite-size rounding of the transition temperature T_c at a first-order phase transition may be estimated from the equation (Imry 1980)

$$C_h^* \sim l/\Delta T_c. \quad (61)$$

It follows from equations (59) and (61) that

$$\Delta T_c / T_c \sim kT / Nl \equiv 1 / N\sigma \quad (62)$$

which is consistent with the results proposed by Imry (1980).

In conclusion, we have shown that the phase transition in the QPM corresponds to the percolation transition of a QBCPM and the thermal properties of the QPM could be derived simply from g_f and g_p . Using the geometrical condition of the phase transition based on such a connection, we have given the geometrical interpretation of the increase of α with q for $q \leq q_c$, changeover from a second-order to a first-order phase transition as q increases (Wu 1982), finite-size scaling of specific heat and the rounding of transition temperature at a first-order phase transition.

The theory presented above may be extended to other interacting systems which have the corresponding correlated percolation or cluster models. The details have been and will be given elsewhere (Hu 1985a, b and unpublished data).

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References

- Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic) pp 46-266
- Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629
- Deutscher G, Zallen R and Adler J (ed) 1983 *Percolation Structures and Processes Ann. Israel Phys. Soc.* vol 5 (Bristol: Adam Hilger)
- Essam J W 1973 *Phase Transitions and Critical Phenomena* vol 2, ed C Domb and M S Green (New York: Academic) pp 197-270
- 1980 *Rep. Prog. Phys.* **43** 833-912
- Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 642
- Hu C K 1983a *Physica* **119A** 609-14
- 1983b *J. Phys. A: Math. Gen.* **16** L321-6
- 1984a *Phys. Rev. B* **29** 5103-8
- 1984b *Phys. Rev. B* **29** 5109-16
- 1984c *Chin. J. Phys. (Taipei)* **22** No 1, 1-12
- 1984d *Chin. J. Phys. (Taipei)* **22** No 4, 1-20
- 1984e *Ann. Rep. Inst. Phys. Acad. Sin. (Taiwan)* **14** 7-12
- 1985a *Chin. J. Phys. (Taipei)* **23** 47-63
- 1985b *Phys. Rev. B* **32** 7325-32
- Hu C K and Kleban P 1981 *Bull. Am. Phys. Soc.* **26** 241
- 1983 *Ann. Rep. Inst. Phys. Acad. Sin. (Taiwan)* **13** 3-22
- Hu C K and Lee H Y 1984 *Chin. J. Phys. (Taipei)* **22** No 3, 1-10
- Imry Y 1980 *Phys. Rev. B* **21** 2042
- Kertész J, Stauffer D and Coniglio A 1983 *Percolation Structures and Processes* ed G Deutscher, R Zallen and J Adler *Ann. Israel Phys. Soc.* vol 5 (Bristol: Adam Hilger) pp 121-47
- Kleban P and Hu C K 1982 *Bull. Am. Phys. Soc.* **27** 92, 325
- Müller-Krumbhaar H 1974 *Phys. Lett.* **50A** 27
- Onsager L 1944 *Phys. Rev.* **65** 117
- Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106
- Reif F 1965 *Fundamentals of Statistical and Thermal Physics* (New York: McGraw-Hill)
- Stauffer D 1979 *Phys. Rep.* **54** 1-74
- Stauffer D, Coniglio A and Adam M 1982 *Adv. Polym. Sci.* **44** 103-58
- Wu F Y 1982 *Rev. Mod. Phys.* **54** 235-68